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SEPARABLE SEMIGROUP ALGEBRAS

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Let K be a commutative ring with identity and let A be a K-algebra. The algebra A is said to be *K-separable* if A is projective over its enveloping algebra $A\otimes_K A^{\mathrm{op}}$. Examples of K-separable algebras include the $n\times n$ matrix algebra $M_n(K)$ with entries in K as well as group algebra KG where G is a finite group with its order invertible in K (see, e.g., [3]). Since the class of K-separable algebras are closed under finite products and that K-separability is invariant under Morita equivalences one sees that the algebra

$$\mathop{\times}_{i=1}^{n} M_{m_i}(KG_i)$$

is K-separable if G_i is a finite group with its order invertible in K.

In this paper we show that every K-separable semigroup algebra KS, S a semigroup, must be of this form. Furthermore we characterize all semigroups S with their semigroup algebras KS separable over K.

In Section 1 we recall some results from semigroup theory. In Section 2 semisimple semigroup algebras are characterized. In Section 3 we prove the main result.

1. Preliminaries

In this section we shall recall some notions as well as results in semigroup theory which are needed in the paper. The interested readers should consult [2] for more complete presentations.

Let S be a semigroup. A *subsemigroup* of S is a nonempty subset of S which is closed under the induced multiplication. If it is a group, then we shall call it a *subgroup* of S. Note that the identity of the subgroup may not be that of S even if the latter exists. An *ideal* I of S is a nonempty subset closed to left and right multiplication by elements of S. In this case one may define a congruence relation $\sim : x\sim y$ if and only if either $x=y$ or both x and y are in I. The factor semigroup

$S/\sim$ is called the *Rees factor semigroup* of S modulo I and is denoted by S/I. A semigroup is *simple* if it does not have any proper ideal.

A *zero element* of S is an element e such that $xe = ex = e$ for every x in S. Thus $\{e\}$ is an ideal of S and is called the *zero ideal.* Note that if I is an ideal, then S/I has a zero element, namely, the congruence class I. A semigroup S with a zero element e is *0-simple* if S has no proper ideal strictly containing e and $S^2 \neq e$. If $S^2 = e$, then S is called a *zero semigroup.*

Lemma 1. *Suppose S is a nonzero semigroup with zero element e. Then S is 0-simple if and only if $SxS = S$ for every nonzero element x of S.*

Proof. We only prove the 'if' part since that is all we need in the paper. Let I be a nonzero ideal of S, and let $a \in I$ such that $a \neq e$. Then $S = SaS \subseteq SIS \subseteq I$ and so $I = S$. Suppose x is a nonzero element of S. Then $S = SxS \subseteq S^2$ and so $S^2 \neq e$ since $S \neq e$.

Let G be a group. We shall denote by G^0 the semigroup obtained by adjoining a zero element to G. Let P be an $n \times m$ matrix with entries in G^0. Then the *Rees matrix semigroup* $\mathcal{M}^0(G; m, n; P)$ is defined to be the set of all $m \times n$ matrices with entries in G^0 such that at most one entry is nonzero. The multiplication is defined by $A \circ B = APB$ for A, B in $\mathcal{M}^0(G; m, n; P)$. Note that the zero matrix is the zero element.

Proposition 2 (Rees [6]). *If S is a finite 0-simple semigroup, then S is isomorphic to $\mathcal{M}^0(G; m, n; P)$ where G is a subgroup of S.*

Remark. If S is a finite simple semigroup, then, by adjoining a zero element to S, one deduces from the above proposition that S is isomorphic to $\mathcal{M}^0(G; m, n; P) - 0$ where 0 denotes the zero matrix. We shall denote this simple semigroup by $\mathcal{M}(G; m, n; P)$.

Let S be a semigroup with zero. An ideal I of S is *0-minimal* if the only ideals of S contained in I are I and the zero ideal.

Lemma 3. *Suppose I is a 0-minimal ideal of S. Then I is either a zero or a 0-simple subsemigroup of S.*

Proof. Suppose $I^2 \neq e$. Since I^2 is an ideal of S contained in I, $I^2 = I$ by the 0-minimality of I. Let x be a nonzero element of I and let $\langle x \rangle$ be the ideal of S generated by x. Then $I = \langle x \rangle$ and so $I = I^3 = I\langle x \rangle I \subseteq IxI \subseteq I$. Therefore I is 0-simple by Lemma 1.

Let S be a semigroup. A *principal series* of S is a finite decreasing sequence of ideals S_i, $i = 1, 2, \ldots, n$, of S

$$S = S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \emptyset$$

such that there is no ideal strictly between S_i and S_{i+1} for $i = 1, 2, \ldots, n$. It is not hard to see that if S has a zero element e, then $S_n = e$ and that every finite semigroup has a principal series. Since each S_i/S_{i+1} is a 0-minimal ideal of S/S_{i+1}, it is either a zero or a 0-simple semigroup by Lemma 3. Here we adopt the convention that $S/\emptyset = S$.

2. Semisimple semigroup algebras

Throughout R will be a ring with identity and S will be a semigroup. We shall denote the semigroup algebra of S over R by RS.

If S is a semigroup with a zero element e then the *contacted semigroup algebra* of S over R is defined by

$$R_0S = RS/Re.$$

If I is an ideal of a semigroup S, then it is easy to show that the contracted semigroup algebra of the Rees factor semigroup S/I over R is simple RS/RI.

Proposition 4 (Munn [5]). *Suppose $S = \mathcal{M}^0(G; m, n; P)$.*

(a) *If D_0S has an identity where D is a division ring, then P is invertible over DG and $m = n$.*

(b) *If P is invertible over RG and $m = n$, then R_0S is isomorphic to the $m \times m$ matrix algebra $M_m(RG)$ over RG.*

Lemma 5. *Let $R \cong \mathsf{X}_j M_{l_j}(D_j)$, a finite product of matrix rings over rings D_j. Let S be a semigroup and let P be an $n \times n$ matrix over RS with entries either* 0 *or elements of S. If P is invertible over each D_jS, then it is so over RS.*

Proof. Observe that if $T \cong T_1 \times T_2 \times \cdots \times T_m$ is a finite product of rings, then $M_l(T) \cong \mathsf{X}_j M_l(T_j)$. This implies that an element of $M_l(T)$ is invertible if and only if its images under the natural projections $\pi_j : M_l(T) \to M_l(T_j)$ are invertible.

Since $RS \cong \mathsf{X}_j [M_{l_j}(D_j)S]$, it is enough to assume that $R = M_l(D)$ and that P is invertible over DS. In this case the result is clear since D can be embedded in $M_l(D)$ via $d \mapsto dI$ where I denotes the identity matrix.

Lemma 6. *Suppose A is a ring not necessarily with an identity and B is an ideal of A. Then A is semiseniple if and only if both B and A/B are. In this case $A \cong B \times A/B$.*

Proof. This follows from the definition of the semisimple algebras and the Wedderburn–Artin theorem.

Proposition 7 (Zel'manov [9]). *If RS is artinian, then R is artinian and S is finite.*

Theorem A. *Let S be a semigroup with a zero element. The following are equivalent.*

(1) *RS is semisimple.*

(2) *R is semisimple and S is finite with a principal series*

$$S=S_1\supset S_2\supset\cdots\supset S_n\supset S_{n+1}=\emptyset$$

such that $S_i/S_{i+1}\cong\mathcal{M}^0(G_i;\, m_i, m_i;\, P_i)$ where G_i is a subgroup of S_i/S_{i+1} with its order invertible in R and where P_i is invertible over RG_i for $i=1,2,\ldots,n-1$.

(3) *R is semisimple and $RS\cong(\mathsf{X}_{i=1}^{n-1} M_{m_i}(RG_i))\times R$.*

Proof. (1)⇒(2). Since RS is semisimple, R is also. Thus $R\cong\mathsf{X}_i\, M_{l_i}(D_i)$ where D_i is a division ring. Now $RS\cong\mathsf{X}_i[M_{l_i}(D_i)S]$ and so $M_{l_i}(D_i)S$ is semisimple. But the latter is isomorphic to $M_{l_i}(D_iS)$ and, thus, is Morita equivalent to D_iS. Hence D_iS is semisimple, and so, by Proposition 7, S is finite. Let $D=D_i$ and consider a principal series of S as shown in (2). By Lemma 6 and induction, $D_0(S_i/S_{i+1})\cong DS_i/DS_{i+1}$ is semisimple for $i=1,2,\ldots,n$. If S_i/S_{i+1} is a zero semigroup, then $D(S_i/S_{i+1})$ is a zero ring, a contradiction. Thus S_i/S_{i+1} is 0-simple and so, by Proposition 2, $S_i/S_{i+1}\cong\mathcal{M}^0(G_i;\, m_i, n_i;\, P_i)$. Since $D_0(S_i/S_{i+1})$ has an identity, P_i is invertible over DG_i and $m_i=n_i$ by Proposition 4(a). Now Lemma 5 implies that P_i is invertible over RG_i.

(2)⇒(3). Proposition 4(b) implies that $R_0(S_i/S_{i+1})\cong M_{m_i}(RG_i)$. Since semisimplicity is invariant under Morita equivalences, $R_0(S_i/S_{i+1})$ is semisimple by Maschke's theorem. Using Lemma 6 we see that RS is semisimple and

$$\begin{aligned} RS&\cong(RS_1/RS_2)\times\cdots\times(RS_{n-1}/RS_n)\times RS_n\\ &\cong R_0(S_1/S_2)\times\cdots\times R_0(S_{n-1}/S_n)\times RS_n\cong\left[\mathop{\mathsf{X}}_{i=1}^{n-1} M_{m_i}(RG_i)\right]\times R. \end{aligned}$$

Note that $S_n=e$, since S has a zero element e.

(3)⇒(1). By Maschke's theorem, RG_i is semisimple. Since semisimple rings are closed under finite products and are invariant under Morita equivalences the result follows.

Remarks. (1) If S has no zero element, then one may adjoin a zero element to S without affecting the semisimplicity of S. The result below may then be deduced from Theorem A.

Theorem A′. *Suppose S is a semigroup without a zero element. Then the following are equivalent.*

(1′) *RS is semisimple.*

(2′) *R is semisimple and S is finite with a principal series*

$$S=S_1\supset S_2\supset\cdots\supset S_n\supset S_{n+1}=\emptyset$$

such that

$$S_i/S_{i+1} \cong \mathcal{M}^0(G_i; m_i, m_i; P_i) \quad \textit{for } i = 1, 2, \ldots, n-1$$

$$\cong \mathcal{M}(G_i; m_i, m_i; P_i) \quad \textit{for } i = n,$$

where G_i is a subgroup of S_i/S_{i+1} with its order invetible in R and where P_i is invertible over RG_i.

(3′) *R is semisimple and $RS \cong \mathsf{X}_{i=1}^{n} M_{m_i}(RG_i)$ where G_i is a finite group with its order invertible in R.*

(2) Theorem A was proved by Munn [5] in case R is a field.

Corollary 8. *Suppose S is a commutative semigroup. Then RS is semisimple if and only if S is a union of finite abelian groups whose orders are invertible in R, and R is semisimple.*

3. K-separable semigroup algebras

Throughout K will denote a commutative ring with identity.

Lemma 9. *Suppose A is a separable K-algebra. Then any left A-module which is K-projective is A-projective.*

Proof. See [3, page 48] or [7, page 13].

Proposition 10. *Suppose A is a finitely generated K-algebra. Then A is K-separable if and only if $A/\mathcal{M}A$ is $K/\mathcal{M}K$-separable for all maximal ideals $\mathcal{M}$ of K.*

Proof. See [3, page 72].

Lemma 11. *Suppose B is an ideal of a ring A with identity. Suppose B is a ring with identity and that A/B is left A-projective. Then $A \cong B \times A/B$ as rings.*

Proof. Consider the exact sequence

$$0 \to B \underset{\mu}{\overset{\pi'}{\leftrightarrows}} A \xrightarrow[\pi]{} A/B \to 0.$$

Since A/B is A-projective π splits and so μ splits. Let π' be a splitting map for μ, and let $\pi'(1) = e$. Then it is not hard to see that B is the left ideal of A generated by e and that e is the identity of B. Thus π' is a ring homomorphism since $\pi'(aa') = aa'e = aea'e = \pi'(a)\pi'(a')$. Hence there exists a ring homomorphism $\phi = A \to B \times A/B$ defined by $\phi(a) = (\pi'a, \pi a)$. It is routine to check that this is indeed an isomorphism.

Proposition 12. *Let R be a ring with identity and let K be a subring contained in the center of R such that R is finitely generated over K. Then $P \in M_n(R)$ is invertible if and only if $\pi_{\mathscr{M}} P \in M_n(R/\mathscr{M}R)$ is invertible for all maximal ideals $\mathscr{M}$ of K. (Here $\pi_{\mathscr{M}} : M_n(R) \to M_n(R/\mathscr{M}R)$ is induced by the canonical projection $R \to R/\mathscr{M}R$.)*

Proof. We only prove the 'left invertible' part, the 'right invertible part' is similar.
Consider the following diagram of functors

$$\begin{tikzcd}
M_n(R)\text{-Mod} \arrow[r, "\approx", "\phi"'] \arrow[d, "F"'] & R\text{-Mod} \arrow[d, "G"] \\
M_n(R/\mathscr{M}R)\text{-Mod} \arrow[r, "\approx", "\phi_{\mathscr{M}}"'] & R/\mathscr{M}R\text{-Mod}
\end{tikzcd} \tag{1}$$

where

$$\phi = R^n \underset{M_n(R)}{\otimes} -, \qquad \phi_{\mathscr{M}} = (R/\mathscr{M}R)^n \underset{M_n(R/\mathscr{M}R)}{\otimes} -,$$

$$F = M_n(R/\mathscr{M}R) \underset{M_n(R)}{\otimes} -, \quad \text{and} \quad G = (R/\mathscr{M}R) \underset{R}{\otimes} -.$$

By the associativity of tensor products, diagram (1) is commutative. It is not hard to show that ϕ and $\phi_{\mathscr{M}}$ are equivalences of categories.

Suppose P is not left invertible over R. Let J be the left ideal of $M_n(R)$ generated by P. Then $A = M_n(R)/J \neq 0$. Therefore $\phi(A) \neq 0$. Since A is finitely generated over $M_n(R)$, $\phi(A)$ is finitely generated over R. Since R is finitely generated over K, so is $\phi(A)$. Now that $\phi(A) \neq 0$, we have that $\phi(A)_{\mathscr{M}} \neq 0$ for some maximal ideal $\mathscr{M}$ of K. By Nakayama's lemma, $(\phi(A)/\mathscr{M}\phi(A))_{\mathscr{M}} = \phi(A)_{\mathscr{M}}/\mathscr{M}\phi(A)_{\mathscr{M}} \neq 0$. Thus

$$G\phi(A) = (R/\mathscr{M}R) \underset{R}{\otimes} \phi(A) \cong \phi(A)/\mathscr{M}\phi(A) \neq 0$$

and so, by the commutativity of (1),

$$0 \neq F(A) = M_n(R/\mathscr{M}R) \underset{M_n(R)}{\otimes} A \cong M_n(R/\mathscr{M}R)/M_n(R/\mathscr{M}R)J.$$

However, $\pi_{\mathscr{M}} P \in M_n(R/\mathscr{M}R)J$ and so $\pi_{\mathscr{M}} P$ is not left invertible over $R/\mathscr{M}R$.
The other direction is obvious.

Theorem B. *Let S be a semigroup with a zero element such that KS has an identity. Then the following are equivalent.*
(1) *KS is K-separable.*
(2) *S is finite with a principal series*

$$S = S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \emptyset$$

such that the Rees factor semigroup S_i/S_{i+1} is isomorphic to the Rees matrix *semigroup* $\mathcal{M}^0(G_i; m_i, m_i; P_i)$ *where* G_i *is a subgroup of* S_i/S_{i+1} *with its order invertible in* K *and where* P_i *is invertible over* KG_i, $i=1,2,\ldots,n-1$.

(3) $KS \cong (\mathsf{X}_{i=1}^{n-1} M_{m_i}(KG_i)) \times K$.

Proof. (1)$\Leftrightarrow$(2). By Proposition 10, KS is K-separable if and only if $(K/\mathcal{M})S \cong KS/\mathcal{M}S$ is $K/\mathcal{M}$-separable for every maximal ideal $\mathcal{M}$ of K. This, in turn, is equivalent to saying that $(K/\mathcal{M})S$ is semisimple for every $\mathcal{M}$. Theorem A, then implies that this is equivalent to $S_i/S_{i+1} \cong \mathcal{M}^0(G_i; m_i, m_i; P_i)$ where G_i has its order invertible in $K/\mathcal{M}$ and where P_i is invertible over $(K/\mathcal{M})G_i$ for every $\mathcal{M}$. If the order of G_i is not invertible in K, then it is contained in a maximal ideal $\mathcal{M}$ of K and so is zero in $K/\mathcal{M}$, a contradiction. Hence, using Proposition 12 with $R=KG_i$, we see that the result follows.

(2)$\Rightarrow$(3). By Proposition 4(b), $K_0(S_i/S_{i+1}) \cong M_{m_i}(KG_i)$ for $i=1,2,\ldots,n-1$, and, since $S_n = e$, $KS_n = K$. Using Lemmas 9 and 11 as well as the implication (2)$\Rightarrow$(1) we see that

$$KS \cong (KS_1/KS_2) \times \cdots \times (KS_{n-1}/KS_n) \times KS_n \cong \left(\mathop{\mathsf{X}}_{i=1}^{n-1} M_{m_i}(KG_i)\right) \times K.$$

Remark. As before, if S does not have a zero element, then one may add a zero element to S without changing the K-separability of KS. As a result one may deduce from the above theorem the necessary and sufficient conditions for KS to be K-separable in case S has no zero element.

Corollary 12. *Let S be a semigroup with a zero element such that $\mathbb{Z}S$ has an identity. Then the following are equivalent.*

(1) *$\mathbb{Z}S$ is $\mathbb{Z}$-separable.*

(2) *S is finite with a principal series*

$$S = S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \emptyset$$

such that $S_i/S_{i+1} \cong \mathcal{M}^0(1; m_i, m_i; P_i)$ *where* P_i *is invertible over* $\mathbb{Z}$.

(3) $\mathbb{Z}S \cong (\mathsf{X}_{i=1}^{n-1} M_{m_i}(\mathbb{Z})) \times \mathbb{Z}$.

Corollary 13. *Let S be a commutative semigroup with a zero element such that $\mathbb{Z}S$ has an identity. Then the following are equivalent.*

(1) *$\mathbb{Z}S$ is $\mathbb{Z}$-separable.*

(2) *S is finite such that every element is an idempotent.*

(3) $\mathbb{Z}S \cong \mathsf{X}_{i=1}^{n} \mathbb{Z}$.

Remark. Shapiro [8] has proved, essentially, Corollary 12. However his proof depends on the fact that $\mathbb{Z}$-projective algebras which are $\mathbb{Z}$-separable are direct products of matrix algebras over $\mathbb{Z}$. This fact was established using the fact that the Brauer group of $\mathbb{Z}$ is zero and that $\mathbb{Z}$ is separably closed.

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